TECHNICAL NOTES

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

No. 231

THE RESISTANCE TO THE STEADY MOTION OF SMALL SPHERES IN FLUIDS

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Washington
February, 1926

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THE RESISTANCE TO THE STEADY MOTION OF SMALL SPHERES IN FLUIDS.*

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Introduction.

There seems to be little reliable information conveniently available as to the resistance encountered by small spheres moving steadily at moderate speeds in fluids. The present paper, while presenting nothing new in the way of either theory or data, has three objects: first, to show that published data are sufficient to furnish approximate information; second, to present this information in form convenient for computation; and, third, to indicate where further research is needed.

* This paper was prepared in the spring of 1924, but the pressure of other work prevented its being put into shape for publication at that time. Meanwhile the work of Liebster and Schiller, which covers the range of values of $\frac{dV\rho}{\mu}$ from 0.12 to 2000, has appeared. Their results, agreeing well with those quoted here, lend confidence to the conclusions drawn in this paper, and also close up the gap $200 < \frac{dV\rho}{\mu} < 2000$. Therefore, it seems worth while to present this paper, especially since it covers a greater range than that covered by Liebster and Schiller, thereby allowing conclusions to be drawn that were not evident from their work; and it shows in a rather striking way how data, apparently unrelated either to each other or to the problem under consideration, may sometimes be utilized with profit by applying to them the principle of dynamical similarity.

Theory

In this paper, the following symbols will be used:

Table I.

Quantity	Symbol	Dimensions
Resistance	D	MLT^{-2}
Diameter of sphere	d	L
Head pressure	$p_h = \frac{D}{\frac{1}{4}\pi d^2}$	$ML^{-1}T^{-2}$
Relative speed - Sphere vs. Medium	V	LT^{-1}
Density of medium	ρ	ML^{-3}
Density of sphere	σ	ML^{-3}
Viscosity of medium	μ	$ML^{-1}T^{-1}$
Mass of sphere	m	M
Length of molecular mean free path	λ	L
Acceleration due to gravity	g	LT^{-2}

Hydrodynamical theory indicates that, if d and V are sufficiently small, and if the fluid may be regarded as (a) homogeneous and (b) infinite in extent, then the flow of fluid round the sphere should be laminar, so that the resistance should be mostly due to the shearing of the medium. In such a case the viscosity μ will be the dominating factor. In fact, for these conditions, the law deduced theoretically by Stokes (Reference 1) seems to have been amply verified by experiment.

This law is

$$D = 3 \pi \mu d V \tag{1}$$

If, on the other hand, d and V are sufficiently large, the resistance should be mostly due to the energy dissipated in the formation of eddies behind the sphere, and hence depend on the kinetic energy of the motion. The law of resistance should then become:

$$D = k \rho d^2 V^2 \tag{2}$$

where k is a constant.

The limits of validity of these two simple laws are not well known, and there is an intermediate range for which no simple mathematical statement of the law of resistance is possible.

We will see if we can not infer some of this information from published data.

The easiest and most reliable way to compare results obtained by different investigators, expressed in different units, is by the dimensional method. The treatment of this method is without the scope of this paper (Reference 2), but our present requirements are quite simple.

If we assume that, with the limitations already mentioned, D_h depends only on d, V, ρ, and μ, then the law of resistance must be of the form

$$f(D_h, d, V, \rho, \mu) = 0 \tag{3}$$

Dimensional reasoning shows that it must be possible to put (3) in the form

$$\frac{D_h}{\rho V^2} = F\left(\frac{dV\rho}{\mu}\right) \qquad D_L = f(R)\,\rho v^2 \qquad (3a)$$

where the form of the function F must be determined by other considerations. For example, in the region where Stokes' law (Equation 1) is valid, the second member of (3a) becomes $\frac{12\mu}{\rho dV}$. In the regime of the square law (Equation 2) it becomes $\frac{k}{\frac{1}{4}\pi}$.

Dimensional reasoning, however, tells us more. Its main result is the so-called "principle of dynamical similarity." In the present case this principle merely says that the value of $\frac{D_h}{\rho V^2}$ should be uniquely dependent on that of $\frac{dV\rho}{\mu}$, - that, if we form the product $\frac{dV\rho}{\mu}$ in any way whatever, we should get the same value of $\frac{D_h}{\rho V^2}$.

To investigate the law of resistance, then, we should determine the form of the function F. To this end, some of the published experimental data have been examined and are shown in Figs. 1-4, where values of $\frac{D_h}{\rho V^2}$ are plotted on logarithmic bases against the corresponding values of $\frac{dV\rho}{\mu}$.

Experimental Data

I. Silvey (Reference 3) observed the rate of fall of drops of mercury, of diameters from 0.012 cm to 0.07 cm (0.005 in. to 0.03 in.) in castor oil, over a range of values of $\frac{dV\rho}{\mu}$ from 0.000024 to 0.0066. His results agree with Stokes' law.

II. Arnold (Reference 4) observed the rate of fall of Rose metal spheres, of diameters from 0.013 cm to 0.14 cm (0.005 in. to 0.055 in.), in colza oil in a tube of internal diameter 1.09 cm (0.43 in.). The values of $\frac{dV\rho}{\mu}$ ranged from 0.002 to 2.4. Since the proximity of the walls of the fall tube caused a violation of condition (b), Page 2 above, the observed speed of fall must be corrected to that which would prevail in an infinite fluid. Such a correction has been proposed by Landenburg (Reference 5). It is

$$V_{inf.} = V_{obs.} \left(1 + 2.4 \frac{d}{d'}\right) \tag{4}$$

where d' is the diameter of the fall tube.

Arnold's observations, corrected by Equation (4), agree with Stokes' law for the smaller values of $\frac{dV\rho}{\mu}$. Not much weight is given to his observations at larger values of $\frac{dV\rho}{\mu}$ since in this range his spheres were so large compared to the size of the fall tube that eddies could not form freely behind the sphere. Hence Stokes' law would tend to hold longer than it would in an infinite fluid.

The results are shown in Figs. 2 and 3, where each plotted point represents the mean of several observations, the observed data being too numerous for convenient individual representation.

III. Allen (Reference 6) observed the motion of air bubbles and of solid spheres, in water and in aniline, over the ranges indicated in Table II.

Table II.

Observations of H. S. Allen

Sphere	Medium	Diameter		$\frac{dV\rho}{\mu}$
		cm	in.	
Air bubble	Aniline	0.007 - 0.11	0.0028 - 0.043	0.009 - 11.0
Air bubble	Water	0.01 - 0.06	0.004 - 0.024	0.270 - 25.0
Paraffin	Aniline	0.07 - 0.30	0.028 - 0.118	0.500 - 20.0
Amber	Water	0.12 - 0.30	0.047 - 0.118	21.000 - 204.0
Steel	Water	0.30 - 0.80	0.118 - 0.315	2300 - 8200

For small values of $\frac{dV\rho}{\mu}$, for which he used air bubbles in aniline, Allen's results are seen to be quite scattered and to fall, in general, below the theoretical curve. The scattering is to be expected from the difficulty in measuring the diameter of the small bubbles (which were caught under a glass and measured microscopically); while the low values obtained for $\frac{Dh}{\rho V^2}$ in this range can be accounted for, in part at least, by the fact that the air in the bubble was gradually frittered away by the medium, - an effect observed by Arnold (Reference 4). Allen's measurements of the diameters are thus too small, and if corrected for this effect, the points would fall nearer the theoretical curve. Allen's observations at small values of $\frac{dV\rho}{\mu}$ can therefore not be regarded as contradictory to Stokes' law.

IV. Liebster and Schiller (Reference 7), observed the

rate of fall of steel spheres, of diameters 0.1 cm to 0.7 cm (0.04 in. to 0.28 in.), in glycerin, sugar solutions, and water. Their observations covered a range of values of $\frac{dV\rho}{\mu}$ from 0.12 to 2000. Their results are shown in Figs. 3 and 4.

V. Millikan and his collaborators have investigated the motion of very small spheres of various materials, - oil, mercury, shellac, etc., - in a number of different gases, at values of $\frac{dV\rho}{\mu}$ less than about 0.0005. It was found that, as long as the ratio of the mean free path of the molecules of the gas to the diameter of the sphere is very small, Stokes' law holds quite well. When, however, this ratio becomes considerable the sphere falls faster than is indicated by Stokes' law. This occurs when λ is large (at reduced gas pressure, for instance), or d is very small, or both, so that condition (a), Page 2 above, is violated. Equation (1) then becomes

$$D = \frac{3\pi\mu d V}{1 + \varphi \frac{\lambda}{d}} \tag{5}$$

The results are summarized and the complete form of the function $\varphi \frac{\lambda}{d}$ is given in a paper by Millikan (Reference 8). Under most of the conditions with which the present paper is concerned, Millikan's correction is quite negligible. Its magnitude is roughly indicated in Table III.

Table III.

Correction to Stokes' Law - Spheres in Air

Pressure		Temp.		Diam. of sphere		$\varphi \cdot \frac{\lambda}{d}$
cm Hg	in. Hg	°C	°F	cm	in.	%
76	29.9	20	68	0.001	0.0004	1.7
76	29.9	20	68	0.0001	0.00004	16.0
38	14.95	20	68	0.001	0.0004	3.3
76	29.9	0	32	0.001	0.0004	1.4
38	14.95	0	32	0.0001	0.00004	30.0

Remarks

The data shown in Figs. 1-4 were obtained by four different investigators using three different kinds of material for the spheres, - solid, liquid, and gaseous. While only liquids were used as resisting media in the experiments there represented, Millikan's experiments verify Stokes' law for motion in gases at small values of $\frac{dV\rho}{\mu}$, except under the conditions noted above. The author has not found data which seem reliable for resistance in gases in the range $0.0005 < \frac{dV\rho}{\mu} < 6000$, but there seems no reason why this should differ from the curve in Figs. 1-4. Hence, while future research should aim to obtain information as to the resistance in gases in the range indicated above, the following conclusions seem justified:

1. The quantities involved in Equation (3), with the limitations indicated on Page 2 above, are the only physical quanti-

ties seriously affecting the resistance to the motion of spheres in fluids in the range $0.00001 < \frac{dV\rho}{\mu} < 6000$.

2. Stokes' law holds fairly accurately for $\frac{dV\rho}{\mu} < 0.5$, and approximately (within about 7%) to $\frac{dV\rho}{\mu} = 1.0$. This means, for the latter case, a drop of water of diameter about 0.008 cm (0.003 in.) falling in air at 20°C (68°F) and 76 cm (29.9 in.) Hg.

3. For the range $0.5 < \frac{dV\rho}{\mu} < 6000$, the law of resistance is given by the curve of Figs. 3 and 4, with a possible error of about 7%.

There are several facts to which it seems hardly necessary to call attention: first, Figs. 1-4 really represent one continuous curve, it being broken up into seven parts in the manner shown for convenience in plotting and computing; second, since $\frac{D_h}{\rho V^2}$ and $\frac{dV\rho}{\mu}$ are dimensionless, any self-consistent set of units can be at once applied to the relation there shown; and, third, the value of the resistance for any particular value of $d^2 \rho V^2$ is finally determined, not by the value of d, V, ρ, or μ separately, nor by that of any incomplete combination of these, - such as dV, or $\frac{dV}{\mu}$, - but only by that of the complete product $\frac{dV\rho}{\mu}$.

Application

We may illustrate the application of the information con-

tained in Figs. 1-4, by computing the terminal speeds of spheres, of various sizes, falling under the influence of gravity. These are obtained by equating the resistance D, to the effective weight of the sphere, $\frac{1}{6}\pi d^3 (\sigma - \rho) g$.

Case I $\left(\frac{dV\rho}{\mu} < 0.5\right)$

$$\frac{1}{6}\pi d^3 (\sigma - \rho) g = 3 \pi \mu d V$$

$$V = \frac{(\sigma - \rho) g}{18 \mu} d^2 \tag{6}$$

Case II $\left(0.5 < \frac{dV\rho}{\mu} < 6000\right)$

In this region we must use the parameters obtained from Figs. 3 and 4. To do this, we proceed as follows:

$$\text{Let} \qquad \frac{dV\rho}{\mu} = x, \ \frac{D_h}{\rho V^2} = y \tag{7}$$

$$\text{i.e.} \quad \frac{D}{\frac{1}{4}\pi d^2 \rho V^2} = \frac{m_g}{\frac{1}{4}\pi d^2 \rho V^2} = \frac{\frac{1}{6}\pi d^3 g (\sigma - \rho)}{\frac{1}{4}\pi d^2 \rho V^2} =$$

$$= \frac{2}{3} \frac{dg (\sigma - \rho)}{\rho V^2} = y \tag{7a}$$

Solving Equations (7) and (7a), for V and d, we get

$$V = \sqrt[3]{\frac{2}{3} \frac{m_g (\sigma - \rho)}{\rho^2} \frac{x}{y}} \tag{8}$$

$$d = \frac{x \mu}{V \rho} \tag{9}$$

Taking simultaneous values of x and y from Figs. 3 and 4, and substituting proper values of μ, ρ, σ, and g, we may compute corresponding values of d and V.

From Equations (6), (8) and (9), have been computed the terminal speeds of spheres of unit specific gravity, falling in air at 20°C (68°F) and 76 cm (29.9 in.) Hg. The results, expressed in c.g.s. units, are shown in Fig. 5.

A b s t r a c t

Data on the resistance to the steady motion of small spheres in fluids, obtained by various investigators, are collected and presented in logarithmic graphs, the dimensionless variables $\frac{Dh}{\rho V^2}$ and $\frac{dV\rho}{\mu}$ being used as coordinates.

The entire range of values of $\frac{dV\rho}{\mu}$ from 0.00001 to 6000, is found to be satisfactorily represented by a single continuous curve.

Stokes' law holds accurately to $\frac{dV\rho}{\mu} = 0.5$ and approximately (within 7%) to $\frac{dV\rho}{\mu} = 1.0$.

The data are applied to the computation of the terminal speed of spheres of unit specific gravity, falling in air.

References

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8. Millikan: Phys. Rev. (2) 22, p.1 (1923).

Resistance of spheres

$0.000012 < \frac{dV\rho}{\mu} < 0.0012$

Silvey: ⊙ Mercury drops in castor oil

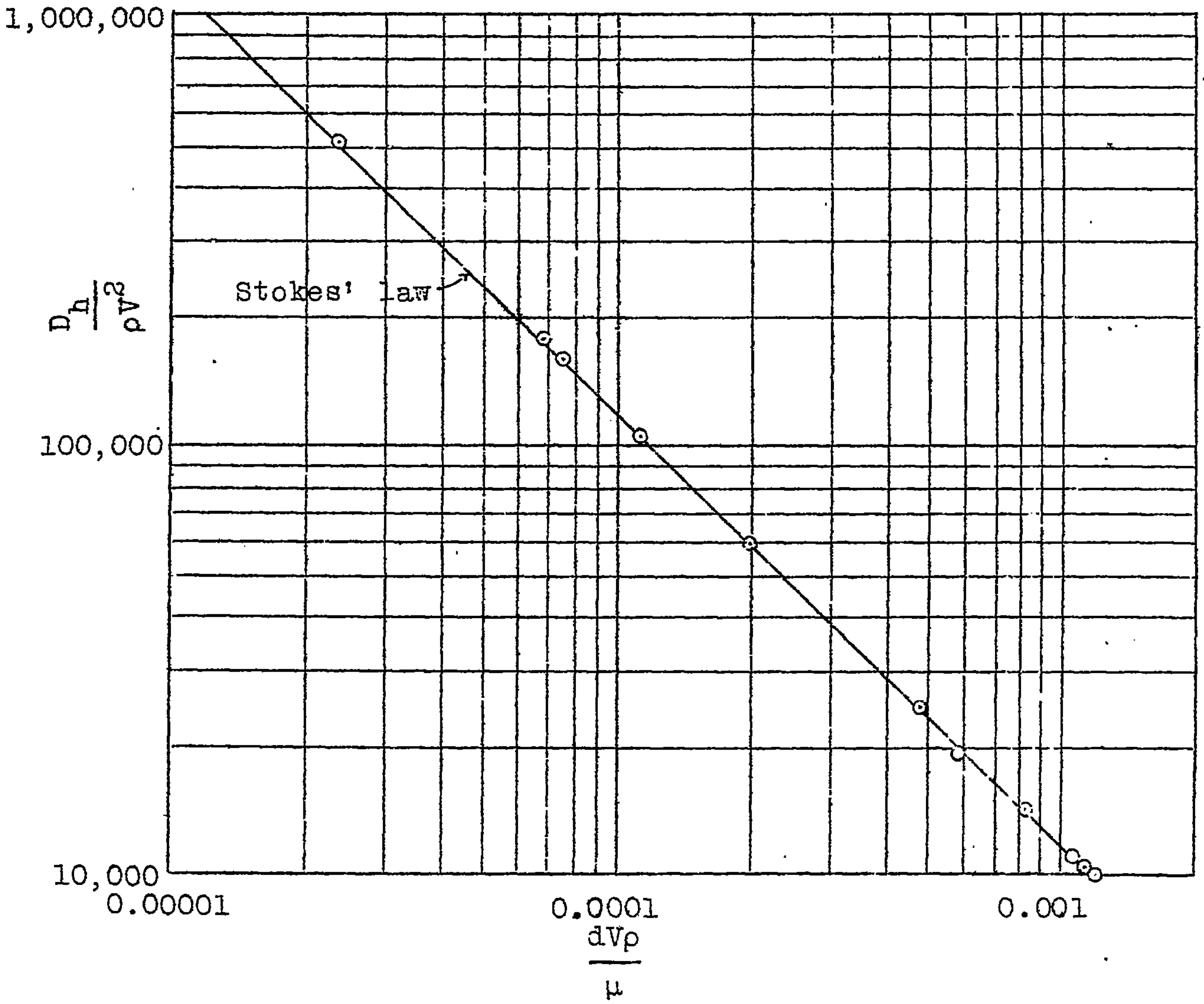

Fig.1

Resistance of spheres

$0.0012 < \frac{dV\rho}{\mu} < 0.12$

Silvey: o Mercury spheres in castor oil.
Arnold: + Rose metal spheres in colza oil.
Allen: x Air bubbles in aniline.

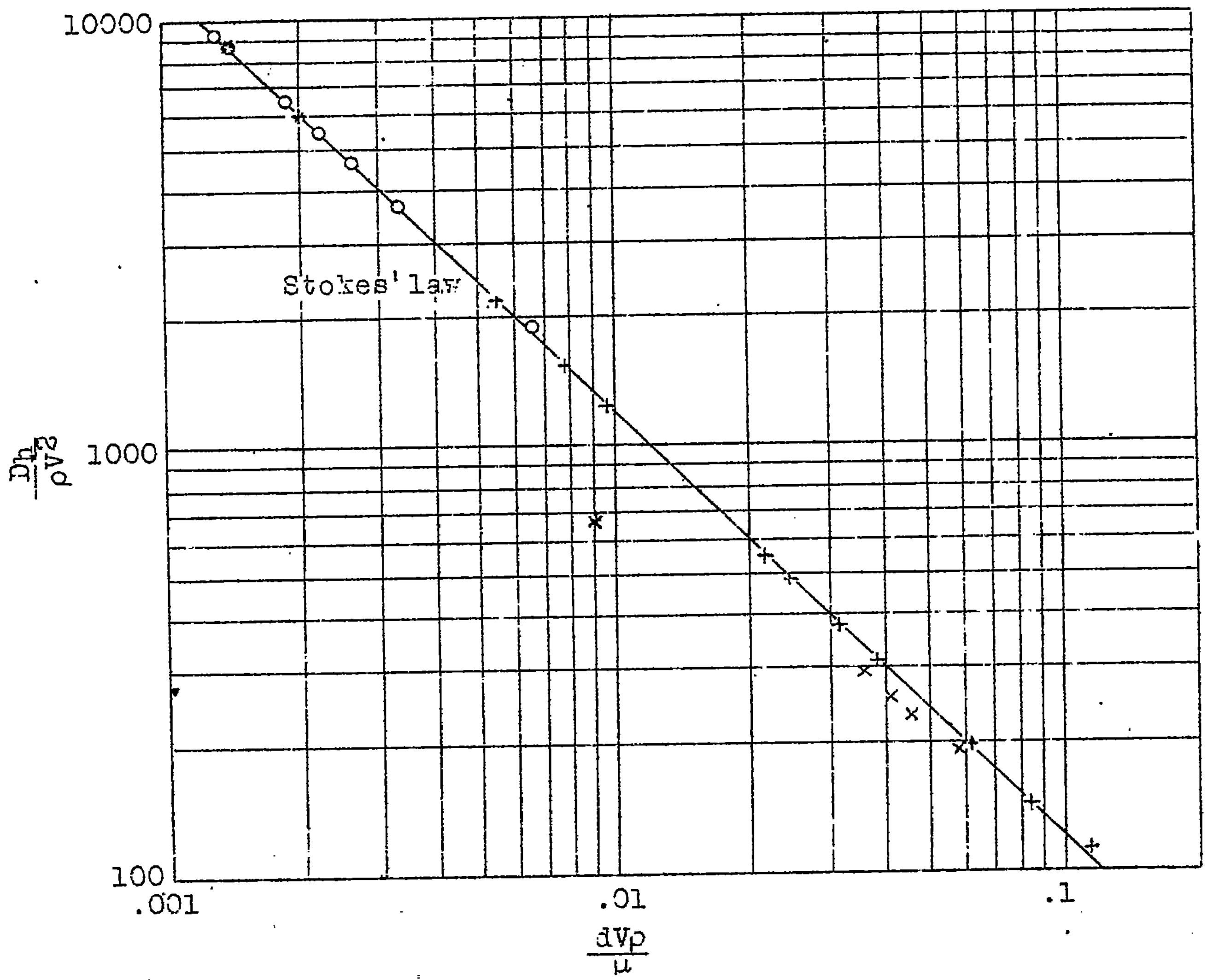

Fig.2

Resistance of spheres. $0.12 < \frac{dV\rho}{\mu} < 10.0$

Arnold:
+ Rose metal spheres in colza oil

Allen:
× Air bubbles in aniline
△ Paraffin spheres in aniline
□ Air bubbles in water

Liebster and Schiller:
○ Steel spheres in glycerine etc.

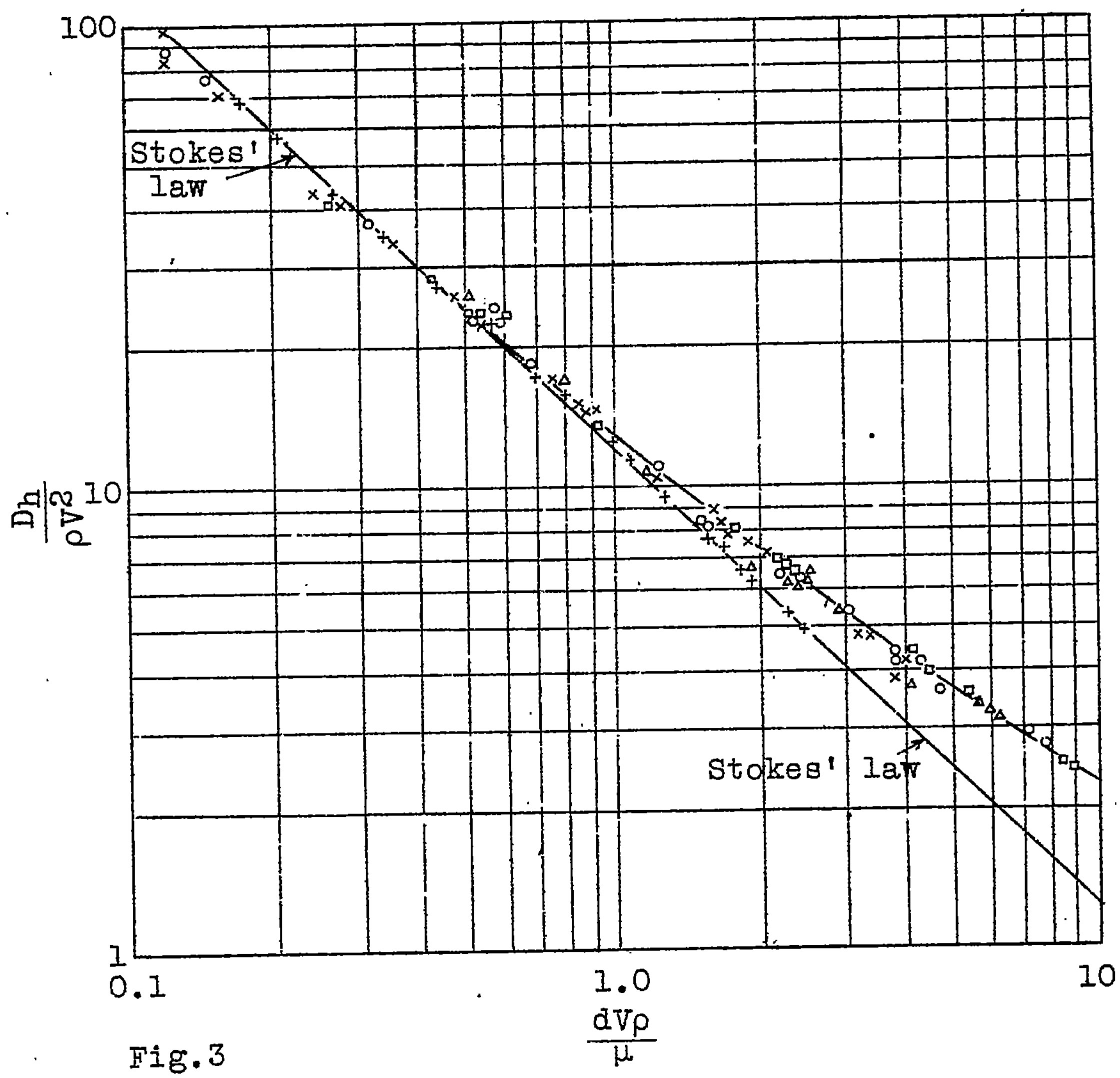

Fig.3

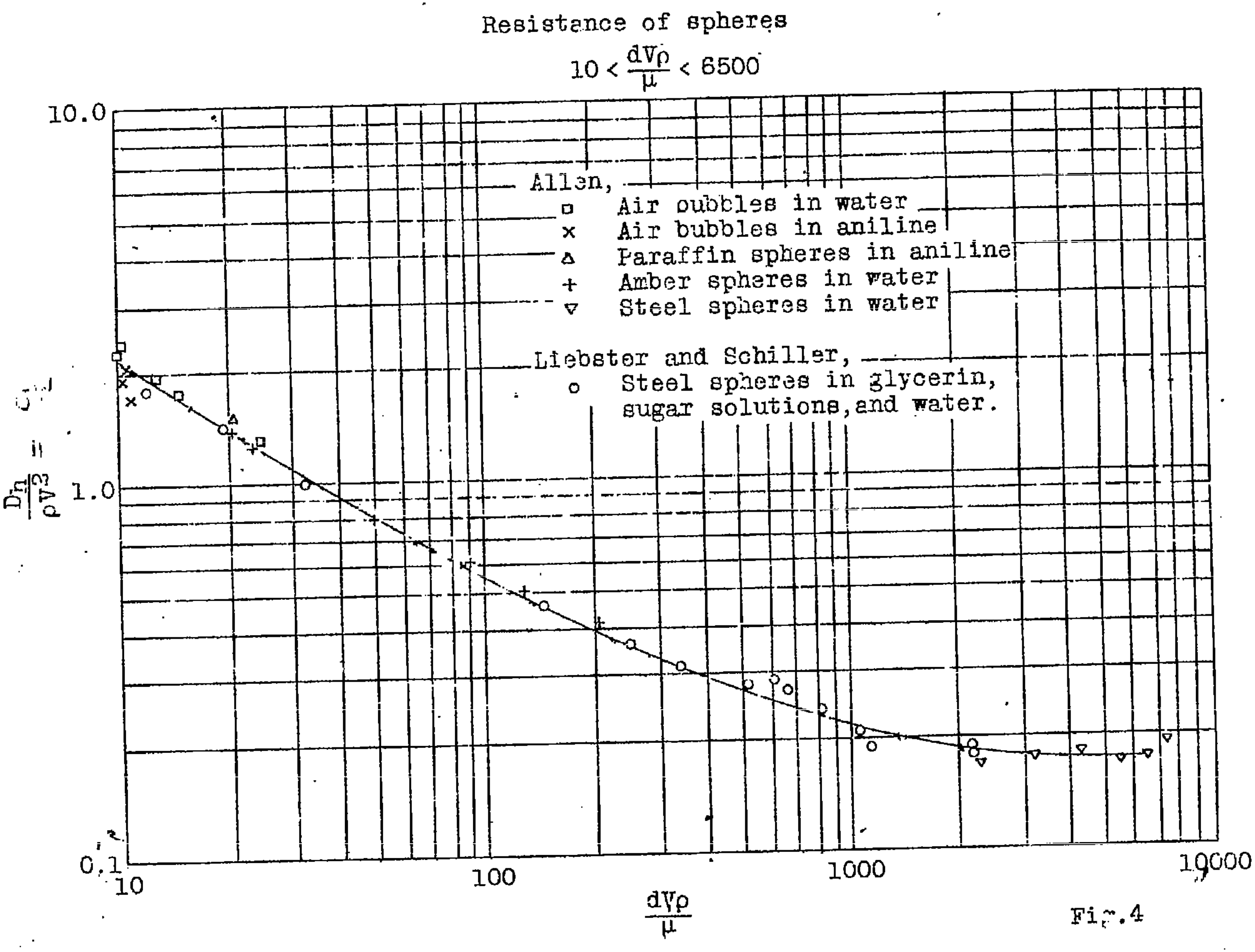
Resistance of spheres
$10 < \frac{dV\rho}{\mu} < 6500$
Allen,
□ Air bubbles in water
× Air bubbles in aniline
△ Paraffin spheres in aniline
+ Amber spheres in water
▽ Steel spheres in water
Liebster and Schiller,
○ Steel spheres in glycerin, sugar solutions, and water.
10.0
1.0
0.1
10
100
1000
10000
$\frac{D\eta}{\rho V^2} = C_W$
$\frac{dV\rho}{\mu}$

Fig.4

Speed of fall.
Spheres of unit specific gravity falling in air at 20°C and 76 cm of Hg.

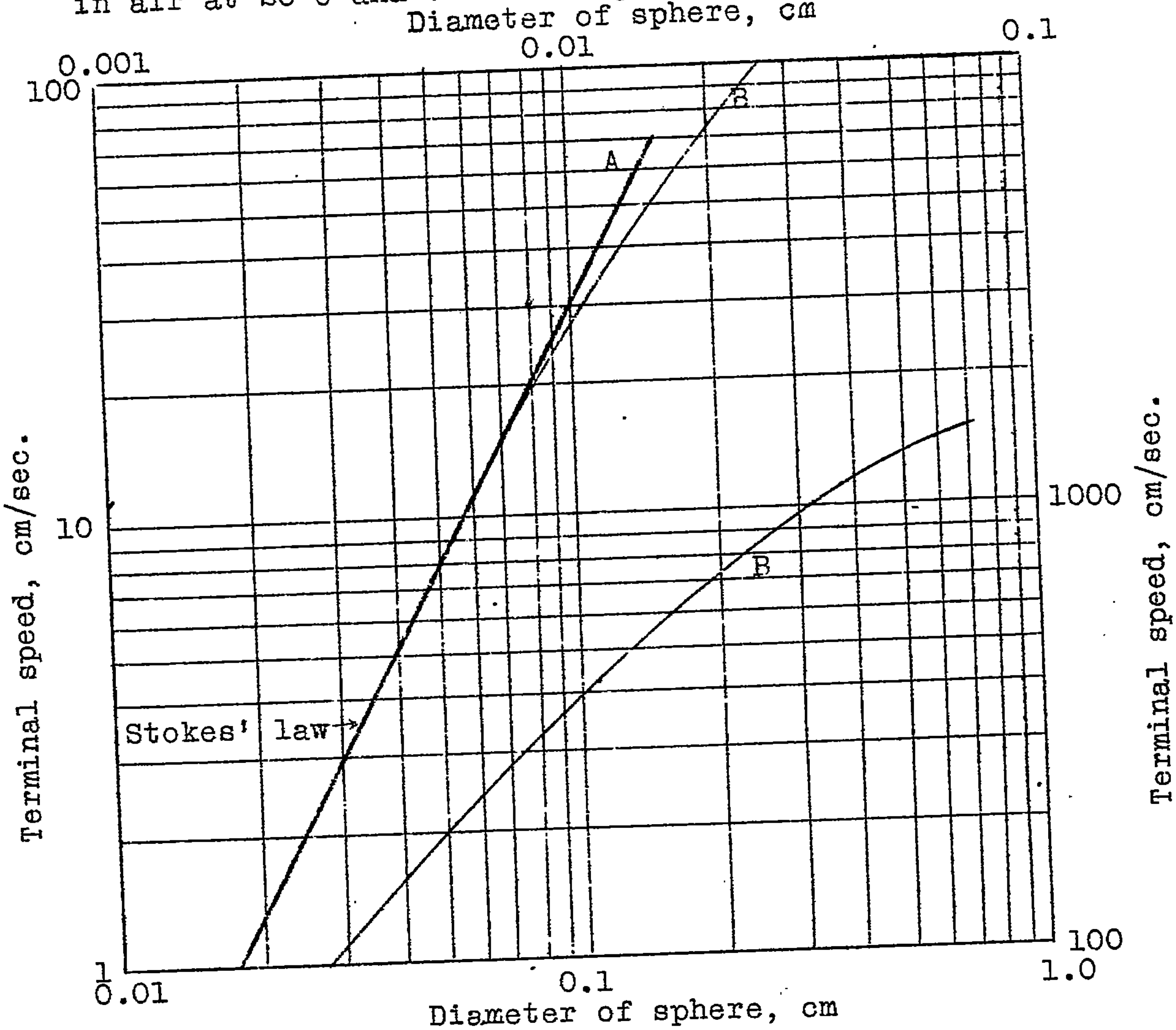

Fig.5